Generalized Microstrip on a Dielectric Sheet

ALBERT L. HOLLOWAY

***Abstract*—This paper describes a procedure for arriving at a close approximation to the capacitance between symmetrically placed conducting strips, possibly of different widths, on opposite sides of a dielectric sheet. The procedure is based on static methods, following Black and Higgins [3] for total capacitance of the structure with vacuum dielectric everywhere, and employing Wheeler's [7] method for determining the series component of dielectric capacitance.**

Dielectric polarization is included. Refraction at the vacuum/dielectric boundary is ignored in the derived method, but its effect is subsequently shown to be small.

The derived equations are valid for all finite impedance, all values of relative dielectric constant, and all conductor widths. The maximum absolute error is estimated to be $0(0.001 \cdot Z')$, where Z' is the impedance of generalized microstrip on a dielectric sheet.

The methods described have general application to open transmission lines on a dielectric sheet, for which the appropriate conformal transformations can be found.

I. Introduction

The PRIMARY application of generalized microstrip is in the design of balun transformers using the tapered microstrip configuration described by Rumsey [1]. Baluns of this type can simultaneously provide both unbalanced to balanced mode conversion and a suitable impedance transformation between an unbalanced coaxial transmission line and the balanced terminals of an antenna or other device. The Klopfenstein [2] impedance taper provides an optimum impedance transformation but requires a detailed knowledge and control of phase velocity and impedance at every point along the taper. The procedures described in this paper are intended to accurately provide those parameters.

The conformal analyses and formulas of this paper are specific to generalized microstrip; however, the method can be applied to a wide varitey of open strip configurations on a dielectric sheet, provided the appropriate conformal transformations can be obtained.

Generalized microstrip in a homogeneous dielectric was analyzed by Black and Higgins [3] using conformal mapping. A modification of their analysis is used in this paper to determine the total capacitance of generalized microstrip in free space. The special case of a conducting strip separated from an infinite conducting plane by a dielectric sheet was analyzed by Dukes [4], [5] with the aid of an electrolytic tank, and later by Wheeler [6]–[8] using approximate conformal mapping and an interpolation technique which produced a correction term to the result obtained by Dukes [4]. More recently Weiss *et al.* [9], [10] developed computer programs treating, as one option, the case of unbalanced microstrip on a dielectric sheet using a numerical method, the dielectric Green's function, described by Sylvester [11].

This paper is concerned with generalized microstrip of the type analyzed by Black and Higgins [3] but on a dielectric sheet as treated, for special cases, by Dukes, Wheeler, and Weiss *et al.* Exact conformal mapping is used, as opposed to approximate conformal mapping employed by Wheeler. Exact methods result in solutions containing special functions and integrals, which are easily computed using standard computer subroutines. A parameter R determines the conductor widths, a/h and b/h. With $R = 0.0$, the conductors are of finite and equal width, while $R = 1.0$ gives a single, finite conducting strip over an infinite ground plane. Values of R between these limits result in two finite-width conducting strips of unequal width.

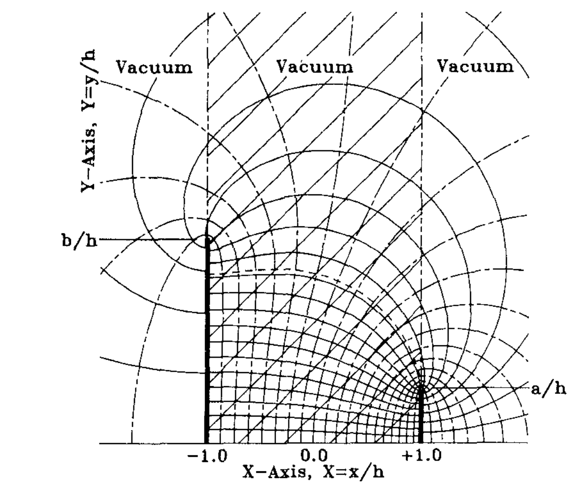
In addition to providing design procedures for generalized microstrip, a further objective is to provide results as free from numerical and analytical approximation as possible. The desired result is an accurate analytic solution to fill the perceived need for a standard static solution suitable for testing and calibrating numerical microstrip solutions.

II. Description of the Transformations

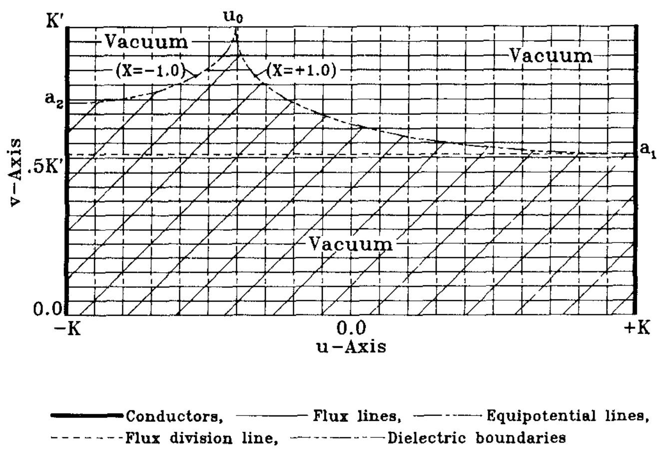
Conformal mapping, and in particular the Schwarz–Christoffel transformation, provides a formalism for transforming a relatively complex geometry, such as that of generalized microstrip, into a simpler geometry where solutions to the two-dimensional Laplace equation can be more easily obtained, usually by inspection. The resulting solutions apply to a transverse cross section of the structure, which is assumed to be constant in the axial direction. Direct application of conformal mapping is restricted to cases with homogeneous dielectric. Two conformal transformations are used in the analysis of generalized microstrip. The first is used to obtain the total capacitance of the conducting strips with vacuum everywhere, following Black and Higgins [3]. The second finds the polarization capacitance of the dielectric due to electrification of the inner faces of the conducting strips, the faces in direct contact with the dielectric sheet. In this second case the dielectric is of finite thickness and may also be of finite width, but since the polarization field is entirely within the dielectric, it is still a homogeneous case. These conformal

Manuscript received July 29, 1987; revised January 19, 1988.
The author is at 3201 Lenox Road N.E., Apt. 30, Atlanta, GA 30324.
IEEE Log Number 8821067.

0018-9480/88/0600-0939$01.00

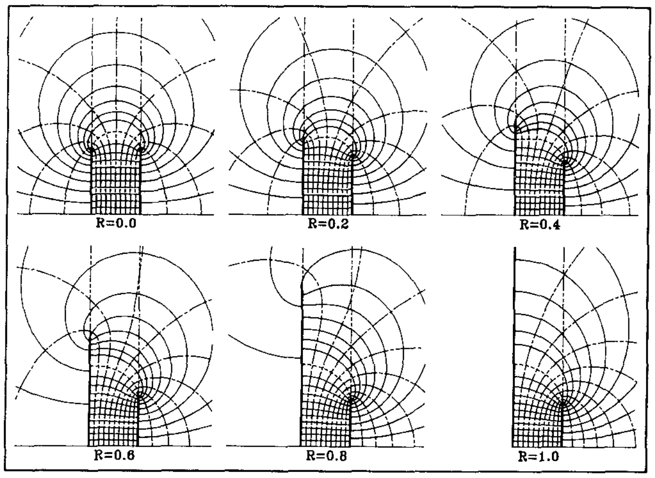


(a)

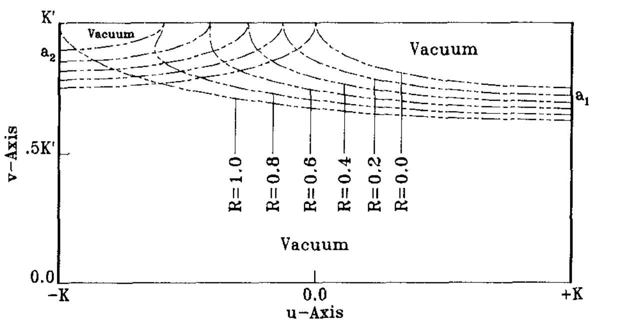


(b)

Fig. 1. Generalized microstrip for $R = 0.6$ and free-space impedance 188.4 Ω. (a) Space coordinates. (b) Flux-potential coordinates.



(a)



(b)

Fig. 2. Generalized microstrip showing the flux, equipotential lines, and dielectric boundaries for values of the parameter R from 0.0 to 1.0 and free-space impedance 94.2 Ω. (a) Space coordinates. (b) Flux-potential coordinates.

TABLE I
DEFINITIONS AND EQUATIONS FOR SPACE AND FLUX-POTENTIAL COORDINATE POINTS OF GENERALIZED MICROSTRIP

	Space plane parameters
a	The half-width of the narrow conductor in space coordinates.
b	The half-width of the wide conductor in space coordinates
h	The half-separation of the conductors in space coordinates
a/h	The ratio of narrow conductor width to conductor separation, Eq (A18)
b/h	The ratio of wide conductor width to conductor separation, Eq (A19)
	Flux-potential plane parameters
a_1	The half-width of the inner face of the narrow conductor, Eq. (A15)
a_2	The half width of the inner face of the wide conductor, Eq (A16)
u_0	The flux-potential image of infinity in space coordinates, Eq (A17)
C_0	The total capacitance of generalized microstrip with vacuum everywhere, Eq (A8).
C_p	The polarization capacitance of the dielectric, Eq (A34)

transformations are derived in Appendixes I and II, respectively.

A. Space and Flux-Potential Coordinates

While it is not necessary to fully understand the derivations of Appendix I to follow the development of the capacitance and impedance equations, it is necessary to understand the nature of the transformed (flux-potential) parameters and their relationship to generalized microstrip in space coordinates. Fig. 8 and Table III, in Appendix I, show the relationships between points in space coordinates and their images in flux-potential coordinates. These relationships are also shown in Figs. 1 and 2, which also show the flux lines, equipotential lines, and dielectric boundaries.

Table I gives definitions of parameters for both the space and flux-potential planes, and, where appropriate, the equation for calculation of the parameter.

B. Space Coordinates

Fig. 1(a) shows the upper half-plane of generalized microstrip with unequal width conductors. The space coordinate plane is normalized by the half-separation of the conductors, such that $X = x/h$ and $Y = y/h$. The conductors are the bold vertical lines at $X = -1.0$ and $X = +1.0$. They are assumed to be at potentials of -1 V and $+1$ V respectively. The thin solid lines between the conductors are the flux lines. The curved short-dashed line, terminating at the edge of the narrow conductor and on the inner face of the wide strip, is the flux division line. All flux below this line terminates on the inner face of the narrow strip, and all flux above terminates on the outer face of the narrow strip. The alternate long–short dashed lines, orthogonal to the flux lines, are the lines of equal potential. The shaded region between the conductors indicates the area that will be occupied by dielectric. The vertical long-double-short dashed lines, bounding the shaded area above the conductors, are the nonmetallized surfaces of the infinitely wide dielectric sheet separating the conductors. In the drawing the shaded area is filled, as indicated, by vacuum and the flux lines shown are those appropriate to vacuum.

C. Flux-Potential Coordinates

Fig. 1(b) is the flux-potential image of the microstrip structure in Fig. 1(a) resulting from the conformal transformation derived in Appendix I. The line coding and other conventions are the same as in space coordinates.

Three points in flux-potential coordinates are of special significance. These are at ($u=K$, $v=a_1$), ($u=-K$, $v=a_2$), and ($u=u_0$, $v=K'$). The point (K, a_1) is the flux-potential image of the point ($X=+1.0$, $Y=a/h$) in space coordinates. The conductor segment at $u=+K$, extending from $v=0.0$ to $v=a_1$, is the flux-potential image of the inner face of the narrow conducting strip in space coordinates. The conductor segment above the point (K, a_1), and extending to the point (K, K'), is the flux-potential image of the outer face of the narrow conducting strip in space coordinates. The point ($-K, a_2$) has the same meaning for the wide conducting strip. The point at (u_0, K') is the flux-potential image of infinity in space coordinates.

The curved long-double-short dashed lines, labeled $X=+1.0$ and $X=-1.0$, are the flux-potential images of the nonmetallized surfaces of the infinite-width dielectric sheet separating the conductors. The dielectric boundaries are transformed between space and flux-potential coordinates using eqs. (A6) and (A7) for the general case. Equations (A23) and (A24) can be used for the special case of balanced microstrip. The shaded regions in both figures are filled by vacuum, but represent the areas that will be filled with dielectric. The flux and equipotential lines shown are those appropriate to free space.

The solid horizontal lines in flux-potential coordinates are flux lines, while the vertical alternate long–short dashed lines are the lines of equal potential. The horizontal short-dashed line at $v=a_1$ is the division line between flux from the inner and outer faces. Flux below this line is from the inner face, while that above is from the outer face. All flux from the outer face is partially in the unshaded region and partially in the shaded region whereas flux from the inner face is entirely within the shaded region.

D. Generalized Microstrip

Fig. 2 shows generalized microstrip for six values of R from 0.0 to 1.0 in space and flux-potential coordinates. The flux and equipotential lines have been omitted in flux-potential coordinates for clarity and readability of the composite graph. All six illustrations are at the same impedance, 94.2 Ω. The dielectric boundaries in flux-potential coordinates are labeled with the appropriate values of R to allow correlation with their space coordinate images. The lines and boundaries are coded as in Fig. 1. Fig. 2 illustrates the changes in the field lines and dielectric boundaries that take place as strip width is changed by variation of R.

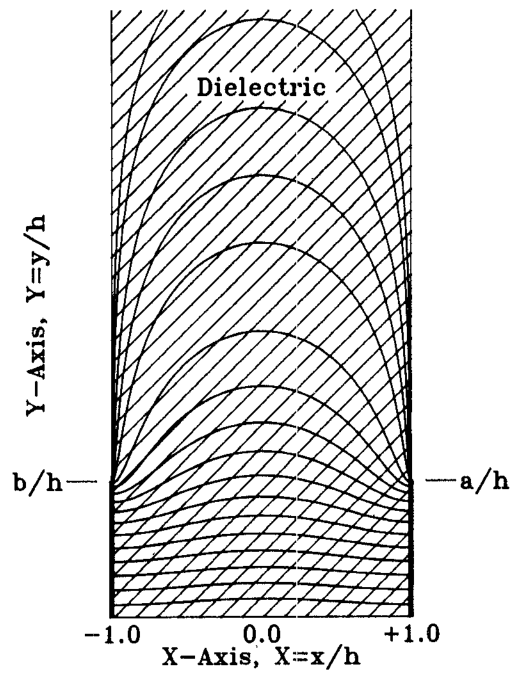


Fig. 3. The polarization field in dielectric for equal-width conducting strips, $R = 0.0$, in space coordinates.

E. Effects of the Dielectric Sheet

The introduction of dielectric with permittivity greater than one into the shaded regions has two effects. The more important effect is that the dielectric is polarized by the electric field, with the polarization capacitance rapidly becoming the dominant capacitance as the relative dielectric constant is increased. The second effect is refraction at the vacuum/dielectric boundary. This effect, of necessity, is not included in the calculations, but its effect is estimated in Section VII.

All illustrations of fields in this paper are either with vacuum everywhere or of the polarization field within the dielectric. Both of these are homogeneous cases. No illustrations are provided for the inhomogeneous case. This omission is necessary because the fields cannot be explicitly calculated for the inhomogeneous case by the methods of this paper.

F. The $\vec{P}$ Field in the Dielectric Sheet

Fig. 3 shows the polarization field within the dielectric in the upper half-plane of space coordinates. Polarization is only present for dielectrics with relative permittivity greater than one.

III. Overview

The objective of the analysis, which follows, is to determine the effective relative dielectric constant, ϵ_{eff}, of generalized microstrip. Relative dielectric constant is defined as: $\epsilon_r = C/C_0$, where C_0 is the vacuum (or geometric) capacitance of the capacitor, and C is its capacitance when filled with some dielectric substance.

If the capacitor is only partially filled with dielectric, the definition can still be used if the capacitance ratio is understood to represent effective dielectric constant.[1] Effective dielectric constant can be written

$$\epsilon_{\text{eff}} = C_d/C_0 \quad (1)$$

with C_d being the capacitance of the capacitor when partially filled with dielectric.

A. Determination of C_d/C_0

The region above a_1, see Fig. 1(b), is inhomogeneous, since flux from the outer faces of the conducting strips is partially in free space and partially in the dielectric region. In this region a direct solution by conformal methods is not possible. The restriction of conformal mapping solutions to homogeneous dielectric is due to the lack of a method for representing refraction of the flux lines at the vacuum/dielectric boundary. In spite of this, it is possible, using conformal mapping techniques, to obtain a very close approximation to the static capacitance of microstrip. The method is to approximate the inhomogeneous dielectric in the flux-potential plane by appropriate subregions, or partial capacitances, connected in parallel or series. Each subregion is filled with either vacuum or homogeneous dielectric. The boundaries of parallel subregions are on flux lines, while those in series have an equipotential boundary. Selection of subregion boundaries in this fashion removes the effect of refraction, which is assumed to be small. The effective dielectric constant is found by combining the capacitances of the vacuum and dielectric subregions in series and parallel and dividing by the total free-space capacitance, C_0.

The derivation of C_d/C_0 is simplified by the use of fractional subregion capacitances of the form c_d/C_0, where c_d is a subregion capacitance. These fractional subregion capacitances can then be combined in series and parallel to form $\epsilon_{\text{eff}} = C_d/C_0$. This procedure is equivalent, but algebraically simpler, than directly combining the subregion capacitances before dividing by C_0 to obtain ϵ_{eff}.

The necessary assumption for application of this method are:

1) The effect of refraction angle at the vacuum/dielectric boundary is negligible.
2) The conductors are of zero thickness.
3) The conductors are lossless.
4) The dielectric sheet is lossless.
5) The longitudinal electric field is negligible.

In addition to the five assumptions above, the design restrictions that $a \ll \lambda_g/2$ and $h \ll \lambda_g/2$ at the highest operating frequency are required to avoid higher modes in nonstatic applications, where a is the width of the narrower strip, h is the dielectric thickness, and λ_g is the wavelength in the dielectric.

[1] Effective dielectric constant is the single relative dielectric constant in all space that produces an equivalent capacitance.

IV. Derivation of Effective Dielectric Constant

It will be necessary, for consistency, to define fractional parameters for conductor width as well as for subregion capacitance. Fractional width parameters are expressed as width over total width, just as fractional capacitance is capacitance over total capacitance. In all cases the capacitances will be "geometric" capacitances, i.e., derived as conductor width over conductor separation, and assumed to be in vacuum.

A. Fractional Parameters

Derivation of the fractional parameters will be made with reference to Fig. 1(b).

1) Fractional Free-Space Width Parameters: The width of the inner face of the narrow conductor, including both half-planes, is $2a_1$ and the total conductor width is $2K'$.

The fractional width of the inner face is

$$A1 = 2a_1/2K' = a_1/K'. \quad (2)$$

Similarly the fractional width of the outer face can be written

$$1 - A1 = 2(K' - a_1)/2K' = 1 - a_1/K'. \quad (3)$$

2) Fractional Free-Space Capacitance Parameters: The capacitance of the inner conductor face is

$$\epsilon_0 2a_1/2K = \epsilon_0 a_1/K.$$

The total capacitance is

$$C_0 = \epsilon_0 2K'/2K = \epsilon_0 K'/K.$$

The fractional capacitance of the inner face is

$$A1 = (\epsilon_0 a_1/K)/(\epsilon_0 K'/K) = a_1/K'. \quad (4)$$

The fractional capacitance of the outer face is

$$1 - A1 = (\epsilon_0 2(K' - a_1)/2K)/(\epsilon_0 K'/K) = 1 - a_1/K'. \quad (5)$$

A consequence of normalization is that the fractional widths, capacitances, and areas are algebraically equivalent. They are not, however, physically equivalent. In the derivation, which follows, the physical usage of these parameters will be indicated at points where confusion might result.

3) Fractional Dielectric Parameters: The two fractional dielectric capacitances will be designated q' and q'', after Wheeler [7]. These two parameters are obtained from two separate conformal transformations. Wheeler termed these parameters filling fractions and defined them as fractional areas. The disignation of q' and q'', as capacitance, is thought to be more physically relevant than area.

a) Derivation of q': The parameter q' is the fractional capacitance of the total dielectric area, when assumed to be purely in parallel.

Let A_d be the area of the dielectric region in the upper half-plane; then the total dielectric area is $2A_d$. The effective width of the dielectric area is $W = 2A_d/S$, where

S = conductor separation = $2K$, giving: $W = A_d/K$. The effective parallel capacitance is $\epsilon_0 W/S = \epsilon_0 W/2K = \epsilon_0 A_d/2K^2$. The fractional capacitance of the total dielectric region is

$$q' = (\epsilon_0 A_d/2K^2)/(\epsilon_0 K'/K) = A_d/2KK'.$$

The area, A_d, is obtained by integrating the area under the curved dielectric boundaries in Fig. 1(b). This gives

$$q' = \frac{1}{2KK'} \int_{-K}^{+K} v(u)\,du \tag{6}$$

where $v(u)$ is the function describing the dielectric boundary. The argument, $v(u)$, is given by eq. (A6) for the general case and by (A24) for the case $R = 0.0$. Use of (A6) requires an iterative procedure. An additional difficulty with the area integration occurs when the dielectric boundary becomes double valued with respect to u, as shown in Fig. 2(b), for $R = 0.8$. When this occurs, special handling is required.

b) Derivation of q'': The second dielectric parameter, q'', is the ratio of the polarization capacitance, C_p, to the total capacitance C_0:

$$q'' = C_p/C_0 \tag{7}$$

where C_p is given by (A34), and $C_0 = \epsilon_0 K'/K$.

The capacitance q', because of its curved dielectric boundary, is actually neither a series nor a parallel capacitance, but a combination of both. It is assumed to be purely parallel for the purpose of comparison with the true parallel capacitance q''.

Fig. 4(a)–(c) shows the behavior of the parallel fractional capacitances, q', q'', and $A1$ as functions of free-space impedance and the parameter R. An important feature is that the true parallel fractional capacitance, q'', is always slightly less than the equivalent parallel fractional capacitance of the total dielectric area, q', for all finite impedance and all R.

All width and capacitance parameters used in the remainder of the main body of this paper are fractional parameters. To avoid unnecessary repetition they will be referred to simply as width, capacitance, or area.

B. Internal and External Capacitances

The total microstrip capacitance can be represented as two capacitances in parallel, designated c_i for the internal capacitance and c_e for the external capacitance. These capacitances are defined as follows:

- c_i = the capacitance resulting from flux terminating on the inner face of the narrow conductor;
- c_e = the capacitance resulting from flux terminating on the outer face of the narrow conductor.

1) Internal Capacitance: The internal capacitance is composed of two partial capacitances in parallel. These are the electric component and the dielectric polarization component. The nature of these capacitances is illustrated by reference to Fig. 1(b). The internal free-space capacitance, given by (4), is $A1$. If the strips are immersed in an infinite, homogeneous dielectric with relative dielectric constant $\epsilon_r > 1$, the internal capacitance can be written

$$c_i = A1(\epsilon_r - 1) + A1. \tag{8}$$

The first term on the right is the dielectric polarization term and the second is the electric, or free-space, term. For the homogeneous case the polarization and electric fields

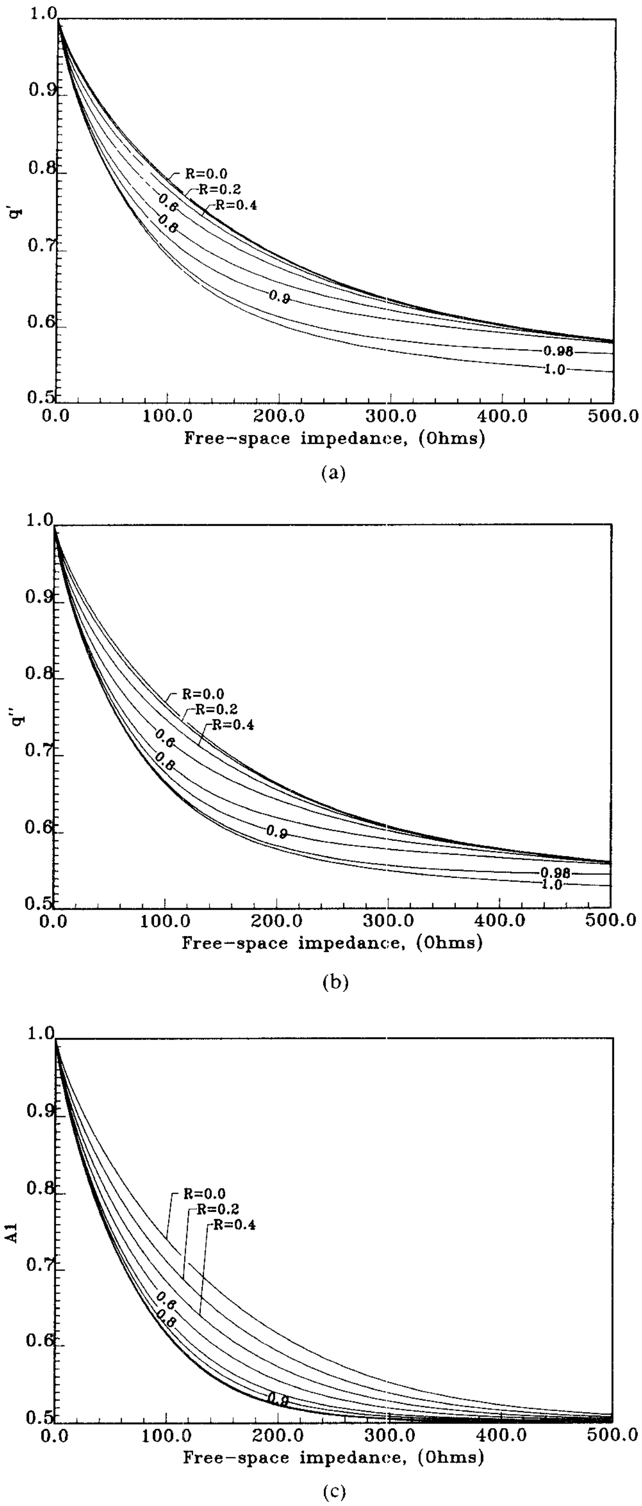


Fig. 4. Values of the parallel capacitances (a) q', (b) q'', and (c) $A1$ for generalized microstrip versus free-space impedance and the parameter R.

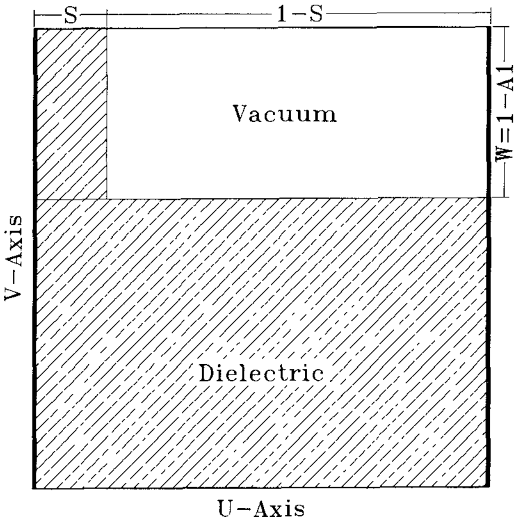


Fig. 5. Schematic of capacitances showing the series external component.

are coincident everywhere, so the internal capacitance can, equivalently, be written as $c_i = \epsilon_r A1$.

The polarization field of interest in microstrip is that shown in Fig. 3. For this case, the polarization field is constrained by the finite dielectric boundaries. The electric and polarization fields are no longer coincident, and the internal capacitance must be expressed as the parallel combination of the polarization and free-space capacitances,

$$c_i = q''(\epsilon_r - 1) + A1 \tag{9}$$

where q'' and $A1$ are the internal polarization and free-space capacitances, respectively. Equation (9) is valid for all values of dielectric constant. For free space it reduces to $c_i = A1$, as it should.

2) External Capacitance: The external capacitance, c_e, is the capacitance resulting from the flux terminating on the outer face of the narrow conductor. This flux fraction is partially in dielectric and partially in air. The lumped equivalent circuit consists of a free-space capacitor in series with a dielectric capacitor. Wheeler [7] noted that the difference $q' - q''$ can be interpreted as the series component of dielectric area. This small surplus area is the series dielectric in the external capacitance.

Fig. 5 is a schematic representation of the internal and external capacitances in parallel. The external capacitance is the series combination of the vacuum and dielectric components, of width $1 - A1$, at the top of the sketch. The width is that given by (3). The shaded region on the upper left, of separation S and width $W = 1 - A1$, represents the series dielectric component of the external capacitance. The unshaded region, with separation $1 - S$, is the series free-space capacitance.

The width of the series dielectric component is $1 - A1$ and its area is $q' - q''$. The series separation S is obtained from

$$W \times S = q' - q'' \tag{10}$$

and

$$S = (q' - q'')/W. \tag{11}$$

The inverse of the external capacitance is

$$1/c_e = 1/(\epsilon_r W/S) + 1/(W/(1-S))$$

giving the external capacitance as

$$c_e = \frac{W}{1 - (1 - 1/\epsilon_r)S}. \tag{12}$$

Substitution of $1 - A1$ for W and the right-hand side of (11) for S in (12) yields the expression for the external capacitance:

$$c_e = \frac{1 - A1}{1 - (1 - 1/\epsilon_r)\dfrac{q' - q''}{1 - A1}}. \tag{13}$$

In the limit $\epsilon_r \to \infty$ the external capacitance reaches a finite limit. This is the maximum possible values of external capacitance:

$$\lim_{\epsilon_r \to \infty} c_e = \frac{1 - A1}{1 - \dfrac{q' - q''}{1 - A1}}. \tag{14}$$

C. Effective Dielectric Constant

Finally, the effective dielectric constant is obtained by combining the internal capacitance, (9), and external capacitance, (13), in parallel:

$$\epsilon_{\text{eff}} = q''(\epsilon_r - 1) + A1 + \frac{1 - A1}{1 - (1 - 1/\epsilon_r)\dfrac{q' - q''}{1 - A1}}. \tag{15}$$

For calculation, substitute a_1/K' for $A1$, with a_1 given by (A16) for the general case and (A26) for $R = 0.0$. Compute q' using (6) and q'' with (7).

In the limit $\epsilon_r \to \infty$ the ratio of effective dielectric constant to relative dielectric constant is

$$\lim_{\epsilon_r \to \infty} (\epsilon_{\text{eff}}/\epsilon_r) = q'' = C_p/C_0. \tag{16}$$

It is noteworthy that the width of the wide strip, $A2$, does not appear in the equation of ϵ_{eff}. This width, b/h in space coordinates, enters the equations via the calculation of q'' and C_0. It also influences the value of q' in the calculation of dielectric area.

D. Approximation of ϵ_{eff}

The most complex procedure in the use of (15) is the computation of q', by numerical integration. Wheeler [7] obtained this integral by approximating the dielectric boundary as an ellipse, which gives good accuracy in the mid-range of impedance for $R = 0.0$ or $R = 1.0$. Because of the wide range of boundary shapes occurring in generalized microstrip (see Fig. 2(b)) this procedure cannot be used in general. The following empirical formula does not require integration and gives Z' within better than ± 0.25

percent of the result from (15):

$$\epsilon_{\text{eff}} \approx q''(\epsilon_r - 1) + A1 + \frac{1 - A1}{1 - 0.536(1 - 1/\epsilon_r)\dfrac{q'' - A1}{1 - A1}}. \tag{17}$$

Equation (17) is valid to the specified accuracy for $0 < Z_0 \leqslant 500\ \Omega$, $0 \leqslant R \leqslant 1$ and $1 \leqslant \epsilon_r$. It is the empirically adjusted average between the case with no dielectric above $A1$, and that where all of the dielectric area $q'' - A1$ is in series.

The impedance, Z', in the presence of the dielectric sheet is given by

$$Z' = Z_0/\sqrt{\epsilon_{\text{eff}}} \tag{18}$$

where Z_0 is the free-space impedance of the microstrip, given by (A9).

The relative phase velocity, v_p, is given by

$$v_p = Z'/Z_0 = 1/\sqrt{\epsilon_{\text{eff}}}. \tag{19}$$

Two other important parameters, a/h and b/h, are computed using (A18) and (A19) in Appendix I. These parameters will usually be computed at the same time as the computationally related parameter a_1. They must be computed prior to calculation of polarization capacitance, since a/h and b/h are arguments of C_p.

V. Comparison to Previous Analyses

The analyses of Dukes [4] and Wheeler [7] are for the special cases $R = 0.0$ and $R = 1.0$; however their formulations of ϵ_{eff} can be directly applied to the general case, using the equations derived in the appendixes.

To compare the ϵ_{eff} equations derived by Dukes and Wheeler, the external capacitance, (13), can be rewritten in the form

$$c_e = (1 - A1)\frac{1}{1 - X} \tag{20}$$

where

$$X = (1 - 1/\epsilon_r)\frac{q' - q''}{1 - A1}. \tag{21}$$

Since X is always less than 1, (20) can be expanded in a binomial series, giving

$$c_e = (1 - A1)(1 + X + X^2 + X^3 + \cdots). \tag{22}$$

The effective dielectric constant can be approximated by using one or more terms of the series in (22) as the external capacitance in parallel with the internal capacitance.

A. Comparison with Dukes [4]

Using only the first term of (22) gives

$$\epsilon_{\text{eff}} \approx q''(\epsilon_r - 1) + A1 + 1 - A1 = q''(\epsilon_r - 1) + 1. \tag{23}$$

This is exactly the equation derived by Dukes [4]. Effectively, (23) simply ignores the presence of dielectric in the external capacitance. Z' computed from (23), and using the exact conformal equations from the appendixes, is accurate to about 1 percent of Z', computed using (15). This shows that the effect of dielectric in the external region is not very significant with respect to impedance.

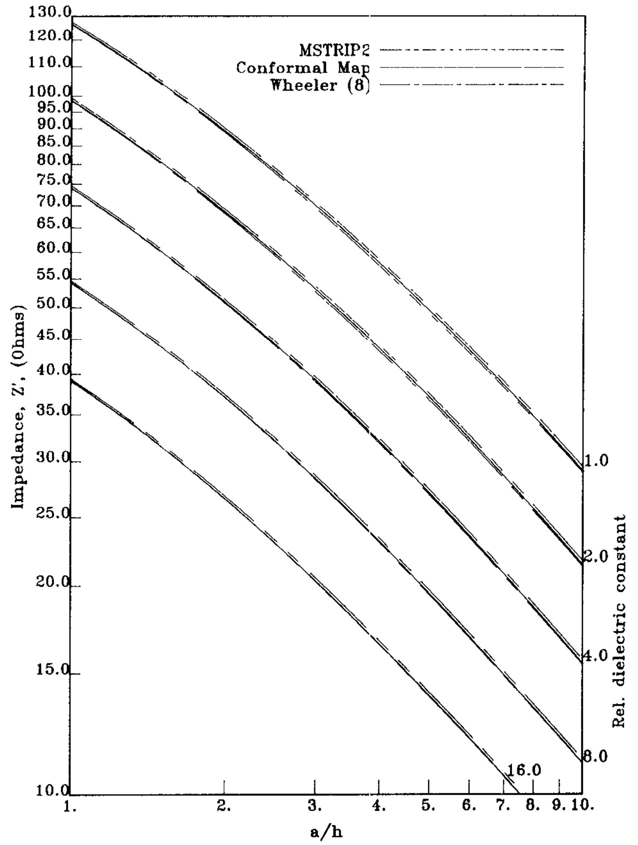


Fig. 6. Microstrip impedance computed by the methods of Wheeler [8], Weiss *et al.* [10], and the conformal methods of this paper versus conductor width and dielectric constant, for a conducting strip on a dielectric sheet over an infinite ground plane.

B. Comparison with Wheeler [7]

Including the first two terms of (22) gives

$$\begin{aligned}\epsilon_{\text{eff}} &\approx q''(\epsilon_r - 1) + A1 + 1 - A1 + (1 - 1/\epsilon_r)(q' - q'') \\ &= q''(\epsilon_r - 1) + 1 + (1 - 1/\epsilon_r)(q' - q'').\end{aligned} \tag{24}$$

Wheeler's [7] expression for ϵ_{eff} consists of two equations. The first is an interpolation to obtain the effective "filling fraction," and the second to obtain the effective dielectric constant. These equations are

$$q = q'' + \frac{q' - q''}{\epsilon_r} \tag{25}$$

and

$$\epsilon_{\text{eff}} = q(\epsilon_r - 1) + 1. \tag{26}$$

Substitution of q from (25) into (26) gives

$$\epsilon_{\text{eff}} = q''(\epsilon_r - 1) + 1 + (1 - 1/\epsilon_r)(q' - q''). \tag{27}$$

Equation (27) is identical to (24), showing Wheeler's solution to be the first two terms of the binomial expansion of the solution derived in this paper.

C. Comparison with Previous Numerical Results

In addition to the analytical comparison with previous static solutions, it is instructive to compare the numerical results to the quasi-static, dielectric Green's function analyses of Sylvester and Weiss *et al.* and to Wheeler's approximate conformal equation. In the conformal analyses the differences result from approximations made by the analyst, while differences between the conformal solution of this paper and the dielectric Green's function result from numerical and discretization errors in computation of the dielectric Green's function and, to a lesser extent, from the omission of refraction in (15).

Fig. 6 is a direct comparison of Z' computed using (15) of this paper, eq. (9) of Wheeler [8], and the MSTRIP2 program of Weiss *et al.* [10]. Both MSTRIP2 and Wheeler [8] differ, at most, from the results of this paper by slightly more than 1 percent, primarily due to approximation, numerical, and discretization errors. Identification of the differences as approximation and numerical errors in Wheeler [8] and Weiss *et al.* [10] rather than as the omission of refraction is made from comparison at $\epsilon_r = 1.0$, where the conformal solution of this paper is exact.

Two more recent papers, on the calculation of microstrip parameters in the presence of a dielectric sheet with $R = 1.0$, were brought to the attention of the author by one of the reviewers. These are Poh *et al.* [12] and Callarotti *et al.* [13]. Both of these papers employ numerical methods for calculation of capacitance. Neither has been compared to the results of this paper.

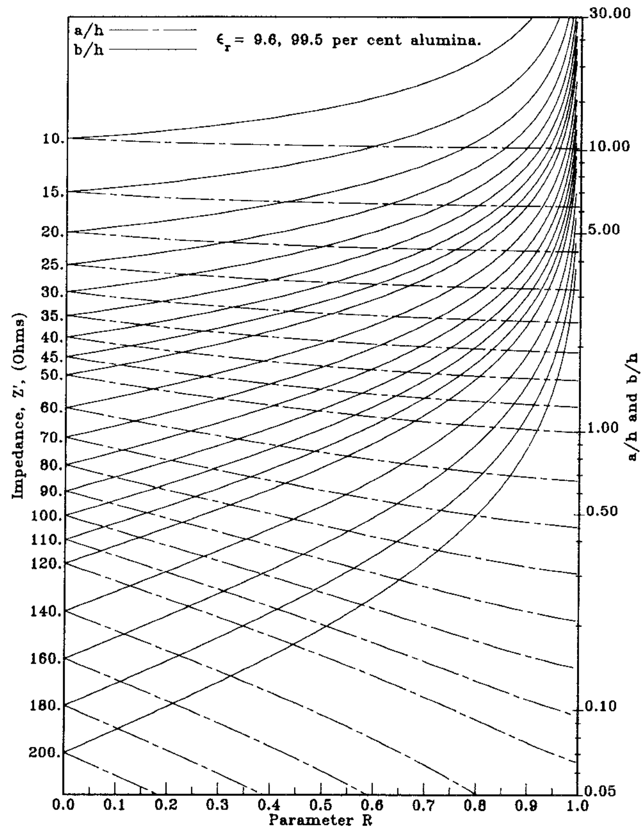


Fig. 7. Conductor widths versus impedance and R for generalized microstrip on alumina.

VI. Computation

For computation of conventional microstrip, eq. (9) of Wheeler [8] is the fastest and simplest method. The 1 percent accuracy provided by this equation is adequate for most practical purposes. For generalized microstrip equation of this paper, (17), provides an accuracy of 0.25 percent. This approximate form is recommended for all but theoretical purposes.

Computation of generalized microstrip begins with selection of values for free-space impedance, Z_0, and the parameter R. A troublesome feature of the parameter R is that its value cannot be selected in advance to provide a particular ratio of strip widths, except for $R = 0.0$ and $R = 1.0$. Another problem is that the impedance in the presence of the dielectric sheet, Z', is not known until the last step, when (18) is evaluated.

Efficient, compact, and accurate computer programs can, nonetheless, be written to provide generalized microstrip design parameters. A simple approach is to construct interpolation tables, such as that plotted in Fig. 7, for alumina.

Table II is a basic calculation schedule for producing such interpolation tables. Separate tables are required for each substrate material. The tables can be stored on disk files and referenced directly by the design program for interpolation. More general programs can be written using iteration. If high accuracy is required, (17) can be used to get within 0.25 percent; (15) can then be used to improve accuracy. Because of the required integration of area, (15) is generally unsuitable for direct use in iterative programs.

TABLE II
A Basic Calculation Schedule for Producing Interpolation Tables, with Subroutines from the IBM 360 Scientific Subroutine Package

	CALCULATION	EQUATION/S	SUB-PROGRAM/S
1	Input Z_0	---------	
2	Compute moduli, k' and k	(A9),(A10),(A11),(A12)	User written
3	Compute elliptic integrals	---------	DCEL1, DCEL2
4	Input R	---------	
5	Compute C1 and C2	(A13),(A14)	User written
6	Compute T_1 and T_2	(A15),(A16)	User written
7	Compute a_1 and a_2	(A15),(A16)	DELI1
8	Compute $E(T_1,k')$ and $E(T_2,k')$	---------	DELI2
9	Compute a/h and b/h	(A18),(A19)	User written
10	Compute C_p	(A33),(A34)	User written, DCEL1
11	Compute $q'' = C_p/C_0$	C_p from above and (A8)	User written
12	Input ϵ_r	---------	
13	Compute ϵ_{eff}	(17)	User written
14	Compute Z'	(18)	User written
15	Compute v_p	(19)	User written

Integration of area and mapping fields between flux-potential and space planes, requires the Jacobian elliptic functions, sn, cn and dn These can be computed using the sub-program DJELF

VII. Estimate of Refraction Error

The total difference between the presence or absence of dielectric in the external region is only about 1 percent of Z' (see Section V-A) and the effect on impedance of refraction at the dielectric boundaries must be much smaller yet. To quantify the error introduced by omission of refraction, a second solution, of equivalent precision to the conformal solution but including refraction, is needed. The difference between these solutions would then approximate the refraction error in the conformal solution.

A. Basis for Refraction Error Estimate

The dielectric Green's function includes refraction, but numerical and discretization errors are so large as to obscure its effect. To obtain a result of sufficient accuracy to permit comparison with the conformal solution of this paper in the range between $\epsilon_r = 1$ and $\epsilon_r \to \infty$, the dielectric Green's function can be asymptotically corrected. Neither the conformal procedures of this paper, nor MSTRIP2, impose a limitation on the upper value of dielectric constant, so any very large value of ϵ_r, within floating point range of the computer, can be used.

The solution of this paper provides the exact values of Z_0 and a/h at $\epsilon_r = 1$. Equation (14) is the maximum value of external capacitance, $c_e$max. It reaches this value at very large dielectric constant. Equation (16) is the value of internal capacitance at very large dielectric constant. Only a fraction of the total external capacitance, $c_e$max, is due to refraction, so the refraction error at large ϵ_r must be very much less than $c_e \max/(\epsilon_r C_p/C_0)$. For $\epsilon_r = 10^{10}$ this ratio is $O(10^{-10})$. From this it is seen that refraction error is negligible at this dielectric constant.

B. Procedure for Calibrating MSTRIP2

The following exact parameters from the conformal solution will be used: I. @$\epsilon_r = 1$, Z_0 (exact), a/h (exact). II. @$\epsilon_r = 10^{10}$, Z' ($\approx$ exact), a/h (exact).

1) With inputs of Z_0 and $\epsilon_r = 1$ find a/h, with the conformal procedure.

NOTE: To obtain values of a/h suitable for precise comparison with MSTRIP2, the value of free-space impedance used in the conformal computation, nominally 376.7 Ω, was set to correspond to the velocity of light used in MSTRIP2, 2.99792458×10^8 m/s.

2) With inputs of $\epsilon_r = 1$ and $w/h = a/h$, find Z with MSTRIP2. Let this be Z_{0_w}.
3) With inputs of Z_0 and $\epsilon_r = 10^{10}$, find Z' with the conformal procedure.
4) With inputs of a/h and $\epsilon_r = 10^{10}$, find Z with MSTRIP2. Let this be Z'_w.
5) Form the ratios $r' = Z_0/Z_{0_w}$ and $r'' = Z'/Z'_w$. (28)
6) Find the values of r for intermediate values of ϵ_r by interpolation in $1/\epsilon_r$ using

$$r = r'' + \frac{r' - r''}{\epsilon_r}. \tag{29}$$

7) With input of the value of a/h used above, and any $1 < \epsilon_r < 10^{10}$ to MSTRIP2, compute a value of Z_w. Adjust the value of Z_w using

$$Z_w(\text{adj.}) = rZ_w. \tag{30}$$

8) With the value of Z_0 used above, and the value of $1 < \epsilon_r < 10^{10}$ used in the preceding step as input to the conformal procedure, compute the value of $Z'(\epsilon_r)$.
9) Compute the percent difference as

$$\text{percent difference} = [Z'(\epsilon_r)/Z_w(\text{adj.}) - 1] \times 100.$$

The same procedure can be used to calibrate the capacitances CAPIE and CAPKE in MSTRIP2, with identical results for the difference in Z'.

C. Refraction Error Results

The value of Z_w(adj.) from MSTRIP2 is always slightly lower than the value of Z' computed by the conformal equations. The maximum difference, of 0.0874 percent, occurs at or near $Z_0 = 111.0$ Ω ($a/h = 1.3183874$) and $\epsilon_r = 3.6$. This difference is tentatively attributed to the omission of refraction effects in the conformal computation of Z'.

The correction technique gives consistent results over the range of Z_0 from 10.0 to 500.0 Ω using MSTRIP2 with $m = 20$. MSTRIP2 does not support computation of generalized microstrip, but the results obtained for a single conducting strip over an infinite ground plane, $R = 1.0$, are thought to be reasonably typical.

VIII. Conclusions

The computational methods used in preparation of this paper have been carefully checked and tested at every stage, and are thought to provide a level of accuracy more than sufficient to support these conclusions.

The exact conformal methods described herein yield static approximations at, or near, the accuracy limit for static techniques. The absolute impedance error is estimated to be $0(0.001 \cdot Z')$, almost entirely attributable to omission of refraction.

The solution has a large parametric range, being valid for all finite impedance, all values of dielectric constant, and all conductor widths. The high accuracy together with the large parametric range makes the solution suitable as a primary static microstrip standard.

Of the five assumptions required, one, that of negligible refraction error, has been shown to be valid. The remaining assumptions, with the possible exception of finite conductor thickness, cannot be removed, or their effect estimated, by conformal techniques.

With reasonable design rules, consistent with the given assumptions and design restrictions, the methods described are capable of producing generalized microstrip designs with an accuracy significantly better than the practical limitations of fabrication and materials. If required, finite conductor thickness and frequency-dependent parameters can be computed by applicable techniques and included as perturbations.

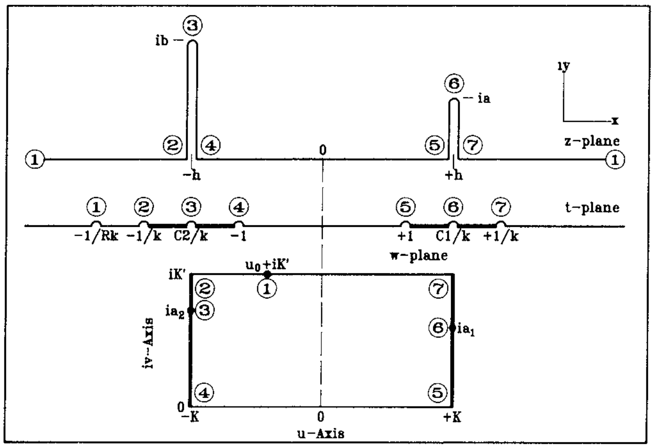


Fig. 8 Schwarz–Christoffel transformation diagram of the z, t, and w planes.

TABLE III
TABULATION OF TRANSFORM POINTS

Point	①	②	③	④	⑤	⑥	⑦
z-plane	Inf.	-h	-h+ib	-h	+h	+h+ia	+h
t-plane	-1/Rk	-1/k	C2/k	-1	+1	C1/k	+1/k
Exponent	-2	-1/2	+1	-1/2	-1/2	+1	-1/2
w-plane	u_0+iK'	-K+iK'	-K+ia_2	-K	+K	+K+ia_1	+K+iK'
t-plane	-1/Rk	-1/k	C2/k	-1	+1	C1/k	+1/k
Exponent	0	-1/2	0	-1/2	-1/2	0	-1/2

APPENDIX I

The following is a summary analysis of the Schwarz–Christoffel transformation diagrammed and tabulated in Fig. 8 and Table III, which applies in the limit $\epsilon_r \to 1$. The solution involves elliptic integrals and functions, with which the reader is assumed to be familiar.

The solution is obtained with respect to a parameter R whose value from 0.0 to 1.0 determines the conductor widths, a/h and b/h.

The notation of elliptic integrals and functions is rather chaotic and nonstandardized, so it was felt desirable to select a single reference and to follow the notation of that reference where possible. The reference selected is the *Handbook of Elliptic Integrals for Engineers and Scientists* by P. F. Byrd and M. D. Friedman [13]. At appropriate points in the following text the notation "B. F. *nnn.nn*" appears following an equation, meaning that the source of the equation or identity is Byrd and Friedman's equation *nnn.nn*.

From Table III the Schwarz–Christoffel equations for the transformations are seen to be

$$dz/dt = D'(t - C1/k)(t - C2/k)(t - 1/Rk)^{-2} \cdot ((t + 1/k)(t - 1/k)(t + 1)(t - 1))^{-1/2} \quad \text{(A1)}$$

and

$$dw/dt = D''((t + 1/k)(t - 1/k)(t + 1)(t - 1))^{-1/2}. \quad \text{(A2)}$$

Dividing (A2) by (A1) yields

$$dz/dw = (dz/dt)/(dw/dt) = D(t - C1/k)(t - C2/k)(t + 1/Rk)^{-2}. \quad \text{(A3)}$$

Equation (A2) is the well-known transformation $t = \text{sn}(w)$, (B.F. 119.01, 129.50), which transforms the interior of a rectangle into a half-plane. Substitution of $\text{sn}(w)$ for t in (A3) gives the form which must be integrated to find $z(w)$:

$$z(w) = D\int^{w} \frac{(\text{sn}(\xi) - C1/k)(\text{sn}(\xi) - C2/k)}{(\text{sn}(\xi) + 1/Rk)^2}\, d\xi \quad \text{(A4)}$$

where $D = D''/D'$.

This map differs from the map of Black and Higgins [3] in the selection of the t-plane mapping points and in the definition of the parameter. The integration is accomplished by the method of undetermined coefficients as outlined in [3]. It results in the general transformation equation:

$$z(w) = \frac{2hk'}{\pi}\left[E(w) + \frac{E' - K'}{K'}Kw + \frac{Rk\,\text{cn}(w)\,\text{dn}(w)}{Rk\,\text{cn}(w) + 1}\right]. \quad \text{(A5)}$$

$X = x/h$ and $Y = y/h$ are obtained in real form by substitution of $z = x + iy$ and $w = u + iv$ in (A5). Equating the real and imaginary parts and after some manipulation, the equations $X = x/h(u, v, k)$ and $Y = y/h(u, v, k)$ are found.

For brevity the following notation will be used:

$$s = \text{sn}(u, k) \qquad c = \text{cn}(u, k) \qquad d = \text{dn}(u, k)$$

$$s_1 = \text{sn}(v, k') \qquad c_1 = \text{cn}(v, k') \qquad d_1 = \text{dn}(v, k')$$

$$A = cdd_1(c_1^2 - k^2s^2s_1^2) \qquad B = ss_1c_1(k^2c^2 + d^2d_1^2)$$

$$C = sd_1 \qquad D = cds_1c_1 \qquad G = k^2scds_1^2$$

$$H = 1 - k^2d^2s_1^2 \qquad P = d^2s_1c_1d_1$$

$$X = x/h = \frac{u}{K} + \frac{2K'}{\pi}\left[E(u, k) - \frac{E}{K}u + \frac{R^2k^2[RkA + (AC - BD)/H]}{(RkC + H)^2 + (RkD)^2} + \frac{P}{G}\right] \quad \text{(A6)}$$

$$Y = y/h = \frac{2K'}{\pi}\left[(v - 1)E' - \frac{R^2k^2[RkB + (AD - BC)/H]}{(RkC + H)^2 + (RkD)^2} + \frac{P}{H}\right]. \quad \text{(A7)}$$

Equations (A6) and (A7) are required for mapping the flux and equipotential lines. Equation (A6), with X set to $+1.0$ and -1.0, together with an integration subroutine, is required for finding the dielectric area in the flux-potential plane. Neither equation is required for the approximate solution, which does not involve calculation of area.

The free-space microstrip capacitance is determined in the flux-potential plane as conductor width divided by conductor separation times the permittivity:

$$C_0 = \epsilon_0 \frac{K'}{K} \quad \text{F/m.} \tag{A8}$$

The free-space impedance, Z_0, is then

$$Z_0 = 376.7 \frac{K}{K'} \quad \Omega. \tag{A9}$$

Given Z_0, the elliptic moduli, k' and k, are easily determined, with a user-written program, by series expansion of the "nome", $q = e^{-(\pi K'/K)}$, which can be expressed in terms of Z_0 as

$$q = e^{-(376.7\pi Z_0)}. \tag{A10}$$

The complementary modulus is then found as

$$k' = [\Theta(0)/\Theta_1(0)]^2, \qquad \text{B.F. } 1052.01 \tag{A11}$$

where

$$\Theta(0) = 1 + 2\sum_1^{\infty}(-1)^m q^{m^2} \quad \text{and} \quad \Theta_1(0) = 1 + 2\sum_1^{\infty} q^{m^2},$$

B.F. 1050.01.

The modulus k is then found as

$$k = \sqrt{1 - k'^2}. \tag{A12}$$

Once the moduli are calculated, the complete elliptic integrals, K, K', E and E', can be calculated using appropriate scientific subroutines. These are functions of impedance only, so they will not need to be recomputed for a given free-space impedance.

In the process of integrating (A4), expressions for $C1 \cdot C2$ and $C1 + C2$ arise. These constants, connecting the w to t and z to t transformations, are

$$C1 \cdot C2 = \frac{E' - R^2 k^2 K'}{R^2 E' - K'} \tag{A13}$$

and

$$C1 + C2 = \frac{R(k^2 K' + K' - 2E')}{k^2(R^2 E' - K')}. \tag{A14}$$

Equations (A13) and (A14) are solved for $C1$ and $C2$ using the quadratic equation.

A. Evaluation at Points in the w Plane

The points a_1, a_2, and u_0 in the w plane are of special significance. They represent, respectively, the half-width of the narrow conductor, the half-width of the wide conductor, and the u coordinate of the flux-potential image of infinity.

The Jacobian form of the incomplete elliptic integrals, with argument $T = \tan\phi = (\sin\phi)/(\cos\phi) = \text{sn}/\text{cn}$, will be used in the derivations that follow. This is the form for the Bulirsch algorithms [14], used in the IBM-360 Scientific Subroutine Package (SSP), which is recommended for these calculations:

$$a_1 = F(T_1, k'), \qquad \text{with } T_1 = \tan\phi_1 = \sqrt{\frac{C1^2 - k^2}{(1 - C1^2)k^2}} \tag{A15}$$

$$a_2 = F(T_2, k'), \qquad \text{with } T_2 = \tan\phi_2 = \sqrt{\frac{C2^2 - k^2}{(1 - C2^2)k^2}} \tag{A16}$$

$$u_0 = F(T_3, k'), \qquad \text{with } T_3 = \tan\phi_3 = -\sqrt{\frac{R^2}{(1 - R^2)}}. \tag{A17}$$

B. Evaluation at Points in the z Plane

The general solution is obtained by evaluating (A5) at point (6), where $z = h + ia$ and $w = K + ia_1$. Similarly, b/h is evaluated at point (3), with $z = -h + ib$ and $w = -K + ia_2$:

$$\frac{a}{h} = \frac{2K'}{\pi}\left[T_1 k(1 - C1^2)\left(\frac{1}{C1} - \frac{R}{RC1 + 1}\right) - E(T_1, k') + \frac{E}{K}F(T_1, k')\right] \tag{A18}$$

$$\frac{b}{h} = \frac{2K'}{\pi}\left[T_2 k(1 - C2^2)\left(\frac{1}{|C2|} - \frac{R}{RC2 + 1}\right) - E(T_2, k') + \frac{E}{K}F(T_2, k')\right]. \tag{A19}$$

C. The Special Case R = 0.0

The general equations simplify significantly when R is taken as zero:

$$X = x/h = \frac{u}{K} + \frac{2K'}{\pi}\left[E(T_u, k) - \frac{E}{K}u + \frac{k^2 scds_1^2}{1 - k^2 d^2 s_1^2}\right] \tag{A20}$$

where $T_u = \text{sn}(u, k)/\text{cn}(u, k)$.

$$Y = y/h = \frac{2K'}{\pi}\left[(v - 1)E' + \frac{d^2 s_1 c_1 d_1}{1 - k^2 d^2 s_1^2}\right]. \tag{A21}$$

In the general case it is not possible to obtain an explicit form $v = v(u, X, R, k)$, but in this case such an expression can be obtained. This expression is very convenient for integrating the dielectric boundary to find the dielectric area in the flux-potential plane.

The expression is obtained by solving (A20) for $s_1 = \sin\phi$ and forming $\tan\phi$. Let

$$S = \frac{\pi}{2K'}\left[X + \frac{u}{K}\right] + u\frac{E}{K} - E(T_u, k) \tag{A22}$$

where $T_u = \text{sn}(u, k)/\text{cn}(u, k)$; then

$$T_v = \tan\phi = \frac{1}{k}\sqrt{\frac{S}{s(scd - sS)}} \tag{A23}$$

giving

$$v(u, X, k) = F(T_v, k) \quad \text{and} \quad V = v/K'. \tag{A24}$$

The two dielectric boundaries are defined by $X = +1.0$ and -1.0 in (A20), but because of symmetry, only one quadrant need be used. Use of (24) allows rapid computation of the area under the curved dielectric boundary in the flux-potential plane.

Substitution of $R = 0.0$ in (A13) and (A14) defines a simpler constant $C0$:

$$C0 = \frac{E'}{K'}. \tag{A25}$$

The conductor widths corresponding to a given impedance are obtained by substitution of $C0$ for $C1$ in (A15). The appropriate argument is

$$a_1 = F(T_0, k'), \qquad \text{with } T_0 = \tan\phi_0 = \sqrt{\frac{C0 - k^2}{1 - C0}}. \tag{A26}$$

For this case, $R = 0.0$, $a_2 = a_1$, $u_0 = 0.0$, and $b/h = a/h$.

The corresponding conductor widths are

$$\frac{a}{h} = \frac{b}{h} = \frac{2K'}{\pi}\left[\frac{T_0 k(1 - C0)}{\sqrt{C0}} - E(T_0, k) + \frac{E}{K}F(T_0, k')\right]. \tag{A27}$$

The equations of this section can also be applied to the case $R = 1.0$ by the method of images.

Appendix II
Analysis of the Dielectric Polarization Capacitance

The analysis of this section is exact for the dielectric polarization due to electrification of the inner faces of the conductors. The result is valid for ϵ_r greater than 1.

Fig. 9 is a diagram of the transformations. The rectangles representing space coordinates (z plane) and flux-potential coordinates (w plane) are mapped to two half-planes using the well-known transformation $t = \text{sn}(z', k)$ and $t' = \text{sn}(w, l)$. These rectangles are of dimensions N by N' and L by L', where N, N', L, and L' are complete elliptic integrals of the first kind with moduli k and l, respectively. The z and z' planes are connected by the scale factor $m = N/n = N'/n'$.

The intermediate half-planes are connected by the bilinear transformation:

$$t' = \frac{(f-d)(t-c)}{(d-c)(f-t)}. \tag{A28}$$

At point (5), $t' = 1/l$, $t = e$, $c = 0$, and $d = \infty$. Substitution of these values in (A28) gives

$$l = d/e. \tag{A29}$$

At point (2), $d = \text{sn}(ma)$. At point (5), $e = \text{sn}(mb + iN') = 1/[k\,\text{sn}(mb)]$, by B.F. 125.01. Substitution for d and e in (A29) gives

$$l = d/e = k\,\text{sn}(ma, k)\,\text{sn}(mb, k). \tag{A30}$$

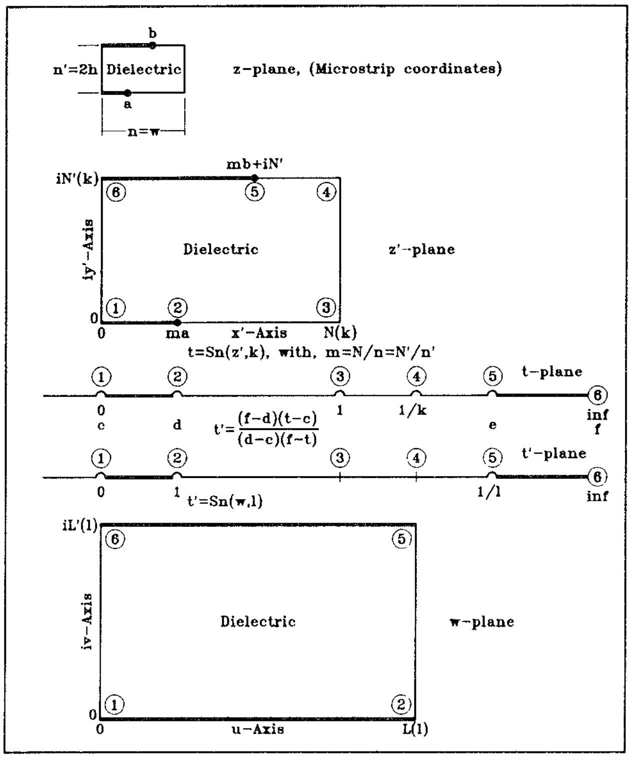


Fig. 9. Transformation diagram for z, t, t', and w planes for determination of the dielectric polarization capacitance of generalized microstrip.

The complete elliptic integral, N, is the width of the dielectric sheet and N' is its thickness, with ma the width of the narrow conductor and mb the width of the wide conductor, and $m = N'/n' = N'/2h$. Then

$$ma = \frac{N'a}{2h} \quad \text{and} \quad mb = \frac{N'b}{2h}.$$

At this point, it is convenient to introduce the assumption that the width of the dielectric sheet is very much greater than its thickness, $2h$, and the width of the narrow conductor, a. This assumption gives results that are independent of dielectric width. If the dielectric is narrow, the exact formulation may be used.

In the limit as $N \to \infty$, $N' = \pi/2$ and $k = 1$. With substitutions for ma and mb, (A30) becomes

$$l = \text{sn}\left(\frac{\pi a}{4h}, 1\right)\text{sn}\left(\frac{\pi b}{4h}, 1\right). \tag{A31}$$

By B.F. 122.09, $\text{sn}(u, 1) = \tanh(u)$, yielding

$$l = \tanh\left(\frac{\pi a}{4h}\right)\tanh\left(\frac{\pi b}{4h}\right). \tag{A32}$$

Equation (A33) is the simplest formulation for finding the elliptic modulus, l, but it may fail numerically for computation at low impedance where l approaches 1. This can be overcome by using the expression for l', which is obtained by substitution of $\tanh^2 u = 1 - \text{sech}^2 u$ and $l^2 = 1 - l'^2$ in (A32), giving

$$l' = \sqrt{\text{sech}^2 A + \text{sech}^2 B - \text{sech}^2 A\,\text{sech}^2 B} \tag{A33}$$

where $A = \pi a/4h$ and $B = \pi b/4h$.

The geometric polarization capacitance, computed by conductor width divided by separation in Fig.9, is

$$C_p = \epsilon_0 \frac{2L(l)}{L'(l)}. \quad \text{(A34)}$$

References

[1] V. H. Rumsey, *Frequency Independent Antennas*. New York: Academic Press, 1966, pp. 11–12.

[2] R. W. Klopfenstein, "A transmission line taper of improved design," *Proc. IRE*, vol. 44, pp. 31–35, Jan. 1956.

[3] K. G. Black and T. J. Higgins, "Rigorous determination of the parameters of microstrip transmission lines," *IRE Trans Microwave Theory Tech.*, vol. MTT-3, pp. 93–113, Mar. 1955.

[4] J. M. C. Dukes, "An investigation into some fundamental properties of strip transmission lines with the aid of an electrilytic tank," *Proc. Inst. Elec. Eng.*, vol. 103, pt. B, pp. 319–333, May 1956.

[5] J. M. C. Dukes, "The application of printed-circuit techniques to the design of microwave components," *Proc. Inst. Elec. Eng.*, vol. 105, pt. B, pp. 155–172, Aug. 1958.

[6] H. A. Wheeler, "Transmission-line properties of parallel wide strips by a conformal-mapping approximation," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-12, pp. 280–289, May 1964.

[7] H. A. Wheeler, "Transmission-line properties of parallel strips separated by a dielectric sheet," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 172–185, Mar. 1964.

[8] H. A. Wheeler, "Transmission-line properties of a strip on a dielectric sheet on a plane," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 631–647, Aug. 1977.

[9] J. A. Weiss, "Microwave propagation on coupled pairs of microstrip transmission lines," *Advances in Microwaves*, vol. 8. New York: Academic Press, 1974, pp. 295–320.

[10] J. A. Weiss, R. S. Withers, and R. C. Lewis, "MSTRIP2: Parameters of microstrip transmission lines and of coupled pairs of lines 1978 Version with its application," MIT Lincoln Laboratory, Tech. Rep. 600, 4 June 1982.

[11] P. Sylvester, "TEM wave properties of microstrip transmission lines," *Proc. Inst. Elec. Eng.*, vol. 115, pp. 43–48, Jan. 1968.

[12] S. Y. Poh, W. C. Chew, and J. A. Kong, "Approximate formulas for line capacitance and characteristic impedance of microstrip line," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-29, pp. 135–142, Feb. 1981.

[13] R. C. Callarotti and A. Gallo, "On the solution of a microstripline with two dielectrics," *IEEE Trans Microwave Theory Tech.*, vol. MTT-32, pp. 333–339, Apr. 1984.

[14] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd ed. New York, Heidelberg, Berlin: Springer-Verlag, 1974.

[15] R. Bulirsch, "Numerical calculation of elliptic integrals and functions," Handbook Series of Special Functions, *Numirische Mathematik*, vol. 7, pp. 78–90, 1965.

Albert L. Holloway was born in Boynton Beach, FL, in 1927. He received the B.E.E. degrees (with high honors) in 1959 and the M.E.E. degree in 1962, both from the University of Florida at Gainesville.

Mr. Holloway is a member of Tau Beta Pi, Phi Kappa Phi, and Pi Mu Epsilon. He holds several U.S. patents in the field of antenna technology.